ELSEVIER Applied Mathematics and Computation 134 (2003) 445–457

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

Stability criteria of impulsive differential systems

A.A. Soliman [1]

Department of Mathematics, Faculty of Sciences, Benha University, Benha 13518, Kalubia, Egypt

Abstract

The notion of eventual stability has been recently discussed.We extend this notion to impulsive systems of differential equations. Our technique depends on Liapunov's direct method.

Keywords: Eventual stability; Uniform eventual stability; Asymptotic eventual stability; Uniform asymptotic eventual stability; Impulsive systems

1. Introduction

In recent years the mathematical theory of impulsive systems of differential equations has been developed by a large number of mathematicians, see e.g. Bainov and Simeonov [1,2], Lakshmikantham et al. [5], and Somoilenko and Perestyuk [7]. Furthermore these systems are adequate mathematical models for numerous processes and phenomena studied in biology, physics technology, etc.

The main purpose of this paper is to extend the notion of eventual stability to impulsive systems of differential equations which is discussed in [6] for systems of ordinary differential equations. The motivation of this work is the recent work of Kulev and Bainov [3]. The paper is organized as follows. In

E-mail address: a_a_soliman@hotmail.com (A.A. Soliman).

[1] Present address: Department of Mathematics, Faculty of Teachers, Al-Jouf, Skaka, P.O. Box 269, Saudi Arabia.

PII: S0096-3003(01)00293-4

Section 2, we introduce some preliminary definitions and results which will be used throughout the paper. In Section 3, we extend the notion of eventual stability to impulsive system of differential equations.

2. Preliminaries

Let $\Re^s_H$ be the s-dimensional Euclidean space with a suitable norm $\|\cdot\|$. Let $\Re^+ = [0,\infty)$. $\Re^s_H = \{x \in \Re^s\colon \|x\| < H\}$.

Consider the system of differential equations with impulses

$$\begin{aligned} &x' = f(t,x) + g(t,y), \quad t \neq \tau_i(x,y), \qquad \Delta x|_{t=\tau_i(x,y)} = A_t(x) + B_t(y), \\ &y' = h(t,x,y), \quad t \neq \tau_i(x,y), \qquad \Delta y|_{t=\tau_i(x,y)} = C_t(x,y), \end{aligned} \tag{2.1}$$

where $x \in \Re^n$, $y \in \Re^m$, $f\colon \Re^+ \times \Re^n_H \to \Re^n$, $g\colon \Re^+ \times \Re^m_H \to \Re^m$, $h\colon \Re^+ \times \Re^n_H \times \Re^m_H \to \Re^m$, $A_t\colon \Re^n_H \to \Re^n$, $B_t\colon \Re^m_H \to \Re^m$, $C_t\colon \Re^n_H \times \Re^m_H \to \Re^m$, $\tau_i\colon \Re^n_H \times \Re^m_H \to \Re^1$.

$$\Delta x|_{t=\tau(x,y)} = x(t+0) - x(t-0), \quad \Delta y|_{t=\tau(x,y)} = y(t+0) - y(t-0).$$

Let $t_0 \in \Re^+$, $x_0 \in \Re^n_H$, $y_0 \in \Re^m_H$. Let $x(t,t_0,x_0,y_0)$, $y(t,t_0,x_0,y_0)$ be solution of the system (2.1), satisfying the initial conditions $x(t_0+0,t_0,x_0,y_0) = x_0$, $y(t_0+0,t_0,x_0,y_0) = y_0$. The solution $(x(t),y(t))$ of the system (2.1) are piecewise continuous functions with points of discontinuity of the first type in which they are left continuous, i.e. at the moment t_i when the integral curve of the solution $(x(t),y(t))$ meets the hypersurface

$$\sigma_t = \{(t,x,y) \in \Re^+ \times \Re^n_H \times \Re^m_H\colon\ t = \tau_i(x,y)\}.$$

The following relations are satisfied:

$$\begin{aligned} &x(t_i-0) = x(t_i), \quad \Delta x|_{t=t_i} = A_t(x(t_i)) + B_t(y(t_i)), \\ &y(t_i-0) = y(t_i), \quad \Delta y|_{t=t_i} = C_t(x(t_i),y(t_i)) \end{aligned}$$

together with system (2.1). We consider the following system with impulses:

$$x' = f(t,x), \quad t \neq \tau_i(x,0), \qquad \Delta x|_{t=\tau_i(x,0)} = A_t(x). \tag{2.2}$$

Let

$$s_t = \{(t,x) \in \Re^+ \times \Re^n_H\colon\ t = \tau_i(x,0)\}.$$

Definition 1. A function $b(r)$ is said to belong to the class $\mathscr{K}$ if $a \in C[\Re^+,\Re^+]$, $b(0) = 0$, and $b(r)$ is strictly monotone increasing in r. Let $\tau_0(x,y) = 0$ for $(x,y) \in \Re^n_H \times \Re^m_H$.

Following [4] we define the sets

$$G_i = \{(t,x,y) \in \Re^+ \times \Re^n_H \times \Re^m_H\colon\ \tau_{i-1}(x,y) < t < \tau_i(x,y)\},$$

$$\Omega_i = \{(t,x) \in \Re^+ \times \Re^n_H\colon\ \tau_{i-1}(x,0) < t < \tau_i(x,0)\}.$$

As in [3], we use the classes $\mathscr{V}_0$ and $\mathscr{W}_0$ of piecewise continuous functions which are analogue to Lyapunov functions.

Definition 2 [3]. We say that the function $V : \Re^+ \times \Re^n_H \times \Re^m_H \to \Re^n_H \times \Re^m_H$ belongs to the class $\mathscr{V}_0$ if the following conditions hold:

1. The function V is continuous in $\bigcup_{i=1}^{\infty} G_i$ and is locally Lipschitzian with respect to x and y in each of the sets G_i.
2. $V(t,0,0) = 0$ for $t \in \Re^+$.
3. For each $i = 1,2,\ldots$ and for any point $(t_0,x_0,y_0) \in \sigma_t$, there exist the finite limits

$$V(t_0 - 0, x_0, y_0) = \lim_{\substack{(t,x,y)\to(t_0,x_0,y_0)\\ (t,x,y)\in G_i}} V(t,x,y),$$

$$V(t_0 + 0, x_0, y_0) = \lim_{\substack{(t,x,y)\to(t_0,x_0,y_0)\\ (t,x,y)\in G_{i+1}}} V(t,x,y),$$

and the equality $V(t_0 - 0, x_0, y_0) = V(t_0, x_0, y_0)$ holds.

4. For any point $(t,x,y) \in \sigma_t$, the following inequality holds:

$$V(t+0, x + A_t(x) + B_t(y), y + C_t(x,y)) \leqslant V(t,x,y). \tag{2.3}$$

Definition 3 [3]. We say that the function W: $I \times \Re^n_H \to \Re^n_H$ belongs to the class $\mathscr{W}_0$ if the following conditions hold:

1. The function W is continuous in $\bigcup_1^{\infty} \Omega_i$ and is locally Lipschitz with respect to x in each of the sets Ω_i.
2. $W(t,0) = 0$ for $t \in \Re^+$.
3. There exist the finite limits

$$W(t_0 - 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\ (t,x)\in\Omega_i}} W(t,x),$$

$$W(t_0 + 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\ (t,x)\in\Omega_i}} W(t,x),$$

and the equality $W(t_0 - 0, x_0) = W(t_0, x_0)$ holds.

4. For any point $(t,x) \in s_t$, the following inequality holds:

$$W(t+0, x + A_t(x)) \leqslant W(t,x). \tag{2.4}$$

Let $V \in \mathscr{V}_0$, and $x(t)$, $y(t)$ be a solution of (2.1) for $(t,x,y) \in \bigcup_1^\infty G_i$. Following [4] we define

$$V_1'(t,x,y) = \lim_{s\to 0} \frac{1}{5}[V(t+s, x+s(f(t,x)+g(t,y)), y+sh(t,x,y)) - V(t,x,y)],$$

and

$$V_{(2.1)}'(t,x,y) = D^+V(t,x,y), \quad t \neq \tau_i(x,y),$$

where $D^+V(t,x,y)$ is the upper right Dini derivative of the function $V(t,x,y)$.

Analogously one can define the function $W_{(2.2)}'(t,x)$ for an arbitrary function $W \in \mathscr{W}_0$ for $(t,x) \in \bigcup_1^\infty \Omega_i$. The following definition is new and related with that of [6].

Definition 4. The zero solution of system (2.1) is said to be eventually stable if for all $\epsilon > 0$, for all $t_0 \in \Re^+$, there exist $\tau_0 > 0$ and $\delta = \delta(t_0, \epsilon) > 0$ for all $(x_0, y_0) \in (\Re_H^n \times \Re_H^m)$ such that

$$\|x_0 + y_0\| < \delta \text{ implies } \|x(t,t_0,x_0,y_0) + y(t,t_0,x_0,y_0)\| < \epsilon, \quad t \geqslant t_0 \geqslant \tau_0,$$

Any eventual stability concepts can be similarly define.

Definition 5. We say conditions (A) hold if the following conditions are satisfied:

(A_1) The functions $f(t,x)$, $g(t,y)$ and $h(t,x,y)$ are continuous in their definition domains, $f(t,0) = g(t,0) = 0$ and $h(t,0,0) = 0$ for $t \in \Re^+$.

(A_2) There exists a constant $L > 0$ such that

$$h(t,x,y) \leqslant L, \quad (t,x,y) \in \Re^+ \times \Re_H^n \times \Re_H^m.$$

(A_3) There exists a continuous function $P: I \to I$ such that $P(0) = 0$ and $\|g(t,y)\| \leqslant P(\|y\|)$ for $(t,x) \in \Re^+ \times \Re_H^n$.

(A_4) The functions A_t, B_t, C_t are continuous in their definition domains and $A_t(0) = B_t(0) = C_t(0,0) = 0$.

(A_5) If $x \in \Re_H^n$ and $y \in \Re_H^n$, then $\|x + A_t(x) + B_t(y)\| \leqslant \|x\|$ and $\|y + c_t(x, y)\| \leqslant \|y\|$, $i = 1, 2$.

(A_6) The functions $\tau_i(x,y)$ are continuous and for $(x,y) \in \Re_H^n \times \Re_H^m$, the following relations hold:

$$0 < \tau_1(x,y) < \tau_2(x,y) < \cdots < \lim_{t\to\infty} \tau_i(x,y) = \infty \quad \text{uniformly in } \Re_H^n \times \Re_H^m$$

and

$$\inf_{\Re_H^n \times \Re_H^m} \tau_{i+1}(x,y) - \sup_{\Re_H^n \times \Re_H^m} \tau_i(x,y) \geqslant \theta > 0, \quad i = 1, 2, \ldots.$$

(A_7) For each point $(t_0, x_0, y_0) \in \Re^+ \times \Re_H^n \times \Re_H^m$, the solution $x(t, t_0, x_0)$, $y(t, t_0, x_0, y_0)$ of the system (2.1) is unique and defined in (t_0, ∞).

(A_8) For each point $(t_0, x_0) \in \Re^+ \times \Re^n_H$, the solution $x(t, t_0, x_0)$ of system (2.2) satisfying $x(t_0 + 0, t_0, x_0) = x_0$ is unique and exists for all $t \in (t_0, \infty)$.

(A_9) The integral curve of each solution of system (2.1) meets each of the hypersurfaces $\{\sigma_t\}$ at most once.

3. Main results

In this section, we give a partial generalization of the work of Kulev and Bainov [3].

Theorem 1. *Assume that*:

(H_1) *Condition* (*A*) *holds.*

(H_2) *There exist functions* $V \in \mathscr{V}_0$, $a \in \mathscr{K}$ *such that*

$$a\|x + y\| \leqslant \|V(t, x, y)\|, \quad (t, x, y) \in \Re^+ \times \Re^n_H \times \Re^m_H.$$

(H_3)

$$V'_{(2.1)}(t, x, y) \leqslant 0 \quad \textit{for } (t, x, y) \in \bigcup_1^\infty G_i.$$

Then the zero solution of the system (2.1) *is eventually stable.*

Proof. Let $0 < \epsilon < H$ and $t_0 \in \Re^+$. Assume that $t_0 \leqslant \tau_1(x, y)$ for $(t, x) \in \Re^n_H \times \Re^m_H$. Since $V(t, 0, 0) = 0$ and from Definition 2, it follows that there exists a $\delta = \delta(t_0, \epsilon) > 0$. Thus it follows that

$$\|x_0 + y_0\| < \delta \text{ implies } \|V(t_0 + 0, x_0, y_0)\| < a(\epsilon).$$

Let $x_0 \in \Re^n_H$, $y_0 \in \Re^m_H$, $\|x_0 + y_0\| < \delta$ and let $x(t) = x(t, t_0, x_0, y_0)$, $y(t) = y(t, t_0, x_0, y_0)$ be a solution of (2.1). From (2.3) and (H_3) it follows that the function $V(t, x, y)$ is monotone decreasing in (t_0, ∞). Then by (H_2) we get

$$a\|x(t) + y(t)\| < V(t, x, y) \leqslant V(t_0 + 0, x_0, y_0) < a(\epsilon), \quad t \geqslant t_0 \geqslant \tau_0,$$

for $t \in (t_0, \infty)$. Therefore $\|x(t) + y(t)\| < \epsilon$. Hence the zero solution of system (2.1) is eventually stable. □

Theorem 2. *Let all conditions of Theorem* 1 *be satisfied except condition* (H_2) *being replaced by*

(H_4)

$$a\|x + y\| \leqslant V(t, x, y) \leqslant b\|x + y\|, \quad a, b \in \mathscr{K}.$$

Then the zero solution of the system (2.1) *is uniformly eventually stable.*

Proof. Since condition (H_4) implies condition (H_2), it follows from Theorem 1 that the zero solution of the system (2.1) is eventually stable. Thus for $\epsilon > 0$, let $\delta = b^{-1}[a(\epsilon)]$ be independent of t_0 for $a, b \in \mathscr{K}$. Let $x_0 \in \Re^n_H$, $y_0 \in \Re^m_H$, $\|x_0 + y_0\| < \delta$ and let $x(t) = x(t, t_0, x_0, y_0)$, $y(t) = y(t, t_0, x_0, y_0)$ be a solution of (2.1).

From (2.3) and (H_3) it follows that the function $V(t, x, y)$ is monotone decreasing in $[t_0, \infty)$. Then by using (H_4) we get

$$a\|x(t) + y(t)\| \leqslant V(t, x, y) \leqslant V(t_0 + 0, x_0, y_0) \leqslant b\|x_0 + y_0\| < b(\delta) < a(\epsilon).$$

Then

$$\|x + y\| < \epsilon \quad \text{whenever } \|x_0 + y_0\| < \delta \text{ for } t \geqslant t_0 \geqslant \tau_0.$$

Thus the zero solution of the system (2.1) is uniformly eventually stable. □

Theorem 3. *Suppose that the assumptions of Theorem* 1 *are satisfied except condition* (H_3) *being replaced by the condition*

(H_5)

$$V'_{(2.1)}(t, x, y) \leqslant -c\|y\| \qquad (t, x, y) \in \Re^+ \times \Re^n_H \times \Re^m_H, \quad c \in \mathscr{K}.$$

We further assume that

(H_6) *There exist functions* $W \in \mathscr{W}_0$ *and* $a_1 \in \mathscr{K}$ *such that*

$$a_1\|x\| \leqslant W(t, x), \quad (t, x) \in \Re^+ \times \Re^n_H.$$

(H_7) *There exist functions* $W \in \mathscr{W}_0$ *and* $c_1 \in \mathscr{K}$ *such that*

$$W'_{(2.2)}(t, x) \leqslant -c_1(W(t, x)), \quad (t, x) \in \bigcup_1^\infty \Omega_i,$$

(H_8)

$$\|W(t, x_1) - W(t, x_2)\| \leqslant \ell\|x_1 - x_2\|.$$

Then the zero solution of the system (2.1) *is asymptotically eventually stable.*

Proof. Since by Theorem 1, we conclude that the zero solution of the system (2.1) is eventually stable. Then we can choose a number $\lambda = \lambda(t_0) > 0$ such that if

$$\|x_0 + y_0\| < \lambda \quad \text{then} \quad \|x + y\| < H, \quad t \geqslant t_0.$$

Now to prove that the zero solution of (2.1) is asymptotically eventually stable, we must show that $\lim_{t\to\infty} \|x(t, t_0, x_0, y_0) + y(t, t_0, x_0, y_0)\| = 0$.

Firstly, we show that $\lim_{t\to\infty} \|y(t, t_0, x_0, y_0)\| = 0$. Suppose that this is not true. Then for some $\epsilon_0 > 0$ there exists a sequence $\{\xi_r\}$ which tends to ∞ for $r \to \infty$ such that

$$\epsilon_0^* \leqslant \|y(\xi_r)\| \leqslant \|y(\xi_r)\|, \quad \text{i.e. } \|y(\xi_r)\| \geqslant \frac{\epsilon_0^*}{\|\phi_0\|} = \epsilon_0, \quad r = 1, 2, \ldots$$

If t_i $(i = 1, 2, \ldots)$ are the moments when the integral curve of the solution $(x(t), y(t))$ meets the hypersurfaces σ_t, then for $t \neq t_i$ by (A_2) we obtain

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \|y(t)\| \right| \leqslant \|y'(t)\| = \|h(t, x(t), y(t))\| \leqslant L.$$

We shall prove that $\|y(t)\| \geqslant \epsilon_0/2$ for $t \in [\xi_r - (\epsilon_0/2L), \xi_r] = J_r$. Let $0 \leqslant \xi_r - t \leqslant \epsilon_0/2L$. Integrating the above inequality from t to ξ_r, we get

$$\int_t^{\xi_r} \frac{\mathrm{d}}{\mathrm{d}t} \|y(\tau)\| \, \mathrm{d}\tau \leqslant L(\xi_r - t) \leqslant \frac{\epsilon_0}{2}.$$

On the other hand, each interval J_r, $r = 1, 2, \ldots$, contains a finite number of points $\{t_s\}$.

As in Theorem 1 of [3], we assume that these points are $t_s, t_{s+1}, \ldots, t_{s+p}$. Then by (A_5) we obtain

$$\begin{aligned}
\int_t^{\xi_r} \frac{\mathrm{d}}{\mathrm{d}\tau} \|y(\tau)\| \, \mathrm{d}\tau &= \int_t^{t_s} \frac{\mathrm{d}}{\mathrm{d}\tau} \|y(\tau)\| \, \mathrm{d}\tau + \sum_{i=s+1}^{s+p} \int_{t_{i-1}}^{t_i} \frac{\mathrm{d}}{\mathrm{d}\tau} \|y(\tau)\| \, \mathrm{d}\tau \\
&\quad + \int_{t_{s+p}}^{\xi_r} \frac{\mathrm{d}}{\mathrm{d}\tau} \|y(\tau)\| \, \mathrm{d}\tau \\
&\geqslant \|y(\xi_r)\| - \|y(t)\|.
\end{aligned}$$

Therefore

$$\epsilon_0 - \|y(t)\| \leqslant \|y(\xi_r)\| - \|y(t)\| \leqslant \int_t^{\xi_r} \frac{\mathrm{d}}{\mathrm{d}\tau} \|y(\tau)\| \, \mathrm{d}\tau \leqslant \frac{\epsilon_0}{2}.$$

Thus we conclude that $\|y(t)\| \geqslant \epsilon_0/2$.

If we choose a suitable subsequence of the sequence $\{\xi_r\}$ which we again denoted by $\{\xi_r\}$, we can assume that the intervals J_r do not intersect one another and $t_0 < \xi_r - (\epsilon_0/2L)$. Then from (H_5) we obtain

$$V'_{(2.1)}(t, x(t), y(t)) \leqslant -c(\epsilon/2) \tag{3.1}$$

in the intervals J_r and

$$V_{(2.1)}(t, x(t), y(t)) \leqslant 0$$

for the remaining values of t for which $(t, x(t), y(t)) \in \bigcup_1^\infty G_i$. Integrating (2.1) and using (2.3), we obtain

$$V(\xi_r, x(\xi_r), y(\xi_r)) \leqslant V(t_0 + 0, x_0, y_0) - c\left(\frac{\epsilon_0}{2}\right)\left(\frac{\epsilon_0}{L}\right) r \to -\infty$$

for $r \to \infty$ which contradicts (H_4). Therefore

$$\lim_{t\to\infty} \|y(t, t_0, x_0, y_0)\| = 0. \tag{3.2}$$

Secondly, we prove that $\lim_{t\to\infty} \|x(t, t_0, x_0, y_0)\| = 0$. To do this we shall prove that $w(t) = W(t, x) \to 0$ for $t \to \infty$. Using (H_8), we get

$$W'_{(2.1)}(t, x) \leqslant W'_{(2.2)}(t, x) + d\|g(t, y)\|,$$
$$t \in \mathfrak{R}^+,\ x \in \mathfrak{R}^n_H,\ y \in \mathfrak{R}^m_H,\ t \neq \tau_i(x, y),\ i = 1, 2, \ldots$$

Thus by (H_7) and (A_3), we have

$$W'_{(2.1)}(t, x(t)) \leqslant -c_1(W(t, x(t))) + dp[\|y(t)\|], \quad t \neq \tau_i(x(t), y(t)). \tag{3.3}$$

We set $\limsup_{t\to\infty} w(t) = \alpha$, $\liminf_{t\to\infty} w(t) = \beta$. Then for an arbitrarily small number $\mu > 0$ we can find sequences $q_n > p_n \to \infty$ for $n \to \infty$ such that $w(p_n) = \beta + \mu$, $w(q_n) = \alpha - \mu$ and $\beta + \mu < w(t) < \alpha - \mu$ for $p_n < t < q_n$. Since the function P is continuous, $P(0) = 0$ and $\lim_{t\to\infty} \|y(t)\| = 0$, it follows that $\lim_{t\to\infty} y(t) = 0$. Thus there exists a positive integer v such that for $n \geqslant v$ and $t \geqslant p_n$ the following inequality holds:

$$P(\|y(t)\|) \leqslant \frac{c_1(\beta + \mu)}{\ell}.$$

Then from (3.3) we have

$$W'_{(2.1)}(t, x(t)) \leqslant -c_1(\beta + \mu) + d\frac{c_1(\beta + \mu)}{\ell} = 0$$

for $n \geqslant \mu$ and $t \in (p_n, q_n)$, $t \neq \tau_i(x(t), y(t))$ which together with (2.4) yields $W(p_n) \geqslant W(q_n)$.

Hence $\beta + \mu \geqslant \alpha - \mu$, which contradicts the assumption that $\alpha > \beta$. This shows that there exists the limit

$$\lim_{t\to\infty} W(t, x(t)) = \gamma \geqslant 0.$$

Now suppose that $\gamma > 0$. Then we can find a number $T > 0$ such that the following inequalities hold:

$$\frac{\gamma}{2} \leqslant W(t, x(t)) \leqslant \frac{3\gamma}{2},$$
$$P(\|y(t)\|) \leqslant \frac{1}{2d} c_1\left(\frac{\gamma}{2}\right)$$

for all $t \geqslant T$. Thus by (3.3), we obtain

$$W'_{(2.1)}(t, x(t)) \leqslant -\frac{1}{2} c_1\left(\frac{\gamma}{2}\right) < 0$$

for $t \geqslant T$. Hence using (2.4) and integrating we obtain

$$W(t,x(t)) = W(T,x(T)) - \frac{1}{2}c_1\left(\frac{\gamma}{2}\right)(t-T) \to -\infty \quad \text{as } t \to \infty,$$

which contradicts (H_6). Therefore

$$\lim_{t\to\infty} \|x(t,t_0,x_0,y_0)\| = 0.$$

Hence the zero solution of the system (2.1) is asymptotically eventually stable. □

Theorem 4. *Let the conditions of Theorems* 2 *and* 3 *be satisfied. Moreover suppose that*

(H_9)

$$W(t,x) \leqslant b_1(\phi_0,x), \qquad (t,x) \in \Re^+ \times \Re^n_H, \quad b_1 \in \mathscr{K}.$$

Then the zero solution of the system (2.1) *is uniformly asymptotically eventually stable.*

Proof. By Theorem 2 it follows that the zero solution of the system (2.1) is uniformly eventually stable. Therefore for any $0 < \epsilon \leqslant M$, there exists $\delta = \delta(\epsilon)$ such that

$$\|x+y\| < \epsilon \quad \text{whenever } \|x_0+y_0\| < \delta, \; t > t_0.$$

Going through as in [3] and Theorem 2, we choose $\delta_1 = \delta_1(\epsilon) > 0$ such that $\delta_1(\epsilon) < \frac{1}{2}\delta(\epsilon)$ and

$$P(s) < \frac{1}{2d}c_1\left(a_1\left(\frac{1}{2}\delta\right)\right) \quad \text{for } 0 \leqslant s \leqslant \delta_1, \tag{3.4}$$

where ℓ is Lipschitz constant for the function W. Moreover let $T_1 = T_1(\epsilon) > 0$ and $T_2 = T_2(\epsilon) > 0$ such that

$$T_1(\epsilon) > \frac{b(M) - a(\frac{1}{2}\delta_1)}{c(\frac{1}{2}\delta_1)}, \tag{3.5}$$

$$T_2(\epsilon) > \frac{2[b_1(M) - a_1(\frac{1}{2}\delta_1)]}{c_1(a_1\frac{1}{2}\delta_1)}. \tag{3.6}$$

Let the positive integer v be such that

$$b(H) - (v-1)\frac{\delta_1 c(\frac{1}{2}\delta_1)}{2L} < 0. \tag{3.7}$$

Let $t_0 \in \Re^+$, $\|x_0+y_0\| < \delta$ and $(x(t),y(t)) = (x(t,t_0,x_0,y_0), y(t,t_0,x_0,y_0))$ be a solution of (2.1). Assume that for all $t \in [t_0, t_0+T_1]$, the inequality $\|y(t)\| \geqslant \frac{1}{2}\delta_1$ holds. Then from (H_5), we obtain

$$V'_{(2.1)}(t,x,y) \leqslant -c\|y(t)\| \leqslant -c\left(\frac{1}{2}\delta_1\right),$$
$$t \in [t_0, t_0 + T_1], \ \ t \neq \tau_i(x(t), y(t)) \ \ i = 1, 2, \ldots \tag{3.8}$$

By integrating (3.8) on $[t_0, t_0 + T_1]$ and making use of (2.3), (H_4), and (3.5) we get

$$\begin{aligned} a\left(\frac{1}{2}\delta_1\right) &\leqslant a\|x(t_0+T_1)+y(t_0+T_1)\| \\ &\leqslant V(t_0+T_1, x(t_0+T_1), y(t_0+T_1)) \quad \text{from } (H_4) \\ &\leqslant V(t_0+0, x_0, y_0) - c\left(\frac{1}{2}\delta_1\right)T_1 \quad \text{from integrating (3.8)} \\ &\leqslant b\|x_0+y_0\| - c\left(\frac{1}{2}\delta_1\right)T_1 \quad \text{from } (H_4) \text{ and (2.3)} \\ &\leqslant b(\delta) - c\left(\frac{1}{2}\delta_1\right)T_1 \\ &\leqslant b(M) - c\left(\frac{1}{2}\delta_1\right)\frac{b(M)-a(\frac{1}{2}\delta_1)}{c(\frac{1}{2}\delta_1)} \quad \text{from (3.5)} \\ &= a\left(\frac{1}{2}\delta_1\right), \end{aligned}$$

which is a contradiction. Thus there exists ξ_1, $t_0 < \xi_1 < t_0 + T_1$, such that

$$\|y(\xi_1)\| < \frac{1}{2}\delta_1, \quad \text{i.e.,} \quad \|y(\xi_1)\| \leqslant \frac{1}{2}\delta_1. \tag{3.9}$$

To prove that for any $t \in [\xi_1, t_0 + T_1 + T_2]$, the inequality $\|y(t)\| < \delta_1 < \frac{1}{2}\delta$ holds, then there exists $\xi_2 \in [\xi_1, t_0 + T_1 + T_2]$ such that $\|x(\xi_2)\| < \frac{1}{2}\delta$. Suppose this is false, then by (H_6), (H_7), and (A_3), in view of (3.4) we obtain

$$W'_{(2.1)}(t,x) \leqslant -\frac{1}{2}c_1\left(a_1\left(\frac{1}{2}\delta\right)\right)$$
$$\text{for } \ t \in [\xi_1, t_0 + T_1 + T_2], t \neq \tau_i(x(t), y(t)). \tag{3.10}$$

By integrating (3.10) on $[\xi_1, t_0 + T_1 + T_2]$ and making use of (H_6), (2.4), (H_9) and (3.6), and going through as in the proof of (3.9), thus there exists $\xi_2 \in [\xi_1, t_0 + T_1 + T_2]$ such that

$$\|x(\xi_2)\| < \frac{1}{2}\delta \quad \text{then} \quad \|x(\xi_2) + y(\xi_2)\| < \frac{1}{2}\delta + \frac{1}{2}\delta < \delta.$$

Now from uniform eventual stability it follows that if

$$\|x(t) + y(t)\| < \epsilon \quad \text{for } t > \xi_2$$

holds, then

$$\|x(t) + y(t)\| < \epsilon \quad \text{for } t > t_0 + T_1(\epsilon) + T_2(\epsilon).$$

Now, let us suppose that there exists $\xi_3 \in [\xi_1, t_0 + T_1 + T_2]$ for which $\|y(\xi_3)\| \geqslant \delta_1$ and let $\xi_5 = \inf\{t \in [\xi_1, t_0 + T_1 + T_2] : \|y(t)\| \geqslant \delta_1\}$. Then $\|y(\xi_5)\| \leqslant \delta_1$ and $\|y(t)\| < \delta$ for $t \in [\xi_1, \xi_5]$. If $\|y(\xi_5)\| < \delta_1$, then from the definition of ξ_5, it follows that $\|y(\xi_5 + 0)\| \geqslant \delta_1$. Hence $\xi_5 = \tau_r(x(\xi_5), y(\xi_5))$ for some positive integer r. But then from (A_5), we obtain that

$$\|y(\xi_5 + 0)\| = \|y(\xi_5) + C_r(x(\xi_5), y(\xi_5))\| \leqslant \|y(\xi_5)\| < \delta_1,$$

which is a contradiction. Thus

$$\|y(\xi_5)\| = \delta_1, \quad \xi_5 \neq \tau_i(x(\xi_5), y(\xi_5)), \; i = 1, 2, \ldots$$

Now using (A_5), we conclude that there exists ξ_4, $\xi_1 < \xi_4 < \xi_5 < t_0 + T_1 + T_2$, such that $\xi_4 \neq \tau_i(x(\xi_4), y(\xi_4))$, $i = 1, 2, \ldots$,

$$\|y(\xi_4)\| = \frac{1}{2}\delta_1 \quad \text{and} \quad \frac{1}{2}\delta_1 < \|y(t)\| < \delta_1 \;\text{ for } t \in (\xi_4, \xi_5)$$

by (A_2), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y(t)\| \leqslant L \quad \text{for } t \neq \tau_i(x(t), y(t)), \; i = 1, 2, \ldots$$

As in the proof of Theorem 3, we obtain that $\xi_5 - \xi_4 \geqslant \delta_1/2L$. Thus by ($H_5$), it follows that

$$V'(t,x,y) \leqslant -c\|y\| \leqslant -c\left(\frac{1}{2}\delta_1\right)$$
$$\text{for } t \in [\xi_4, \xi_5], \; t \neq \tau_i(x(t), y(t)), \; i = 1, 2, \ldots \tag{3.11}$$

Also by integrating (3.11) and making use of (2.3), we obtain

$$\begin{aligned} V(\xi_5, x(\xi_5), y(\xi_5)) &\leqslant V(\xi_4, x(\xi_4), y(\xi_4)) - c\left(\frac{1}{2}\delta_1\right)(\xi_4 - \xi_5) \\ &\leqslant V(\xi_4, x(\xi_4), y(\xi_4)) - c\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}. \end{aligned}$$

Thus we have proved that if $\|x_0 + y_0\| < \delta$, the following two cases are possible:
1. $\|x(t) + y(t)\| < \epsilon$, $t \geqslant t_0 + T_1 + T_2$, or
2. there exist ξ_4, ξ_5, $t_0 < \xi_4 < \xi_5 < t_0 + T_1 + T_2$, such that

$$V(\xi_5, x(\xi_5), y(\xi_5)) \leqslant V(\xi_4, x(\xi_4), y(\xi_4)) - c\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}.$$

In the same way we prove that one of the following two possibilities takes place:
1. $\|x(t)+y(t)\|<\epsilon,\ t\geqslant t_0+2[T_1(\epsilon)+T_2(\epsilon)]$, or
2. there exist $\xi_9, \xi_{10},\ t_0+T_1+T_2<\xi_9<\xi_{10}<t_0+2[T_1+T_2]$, such that

$$V(\xi_{10},x(\xi_{10}),y(\xi_{10}))\leqslant V(\xi_9,x(\xi_9),y(\xi_9))-c\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}.$$

By induction we can prove that if $\|x(t)+y(t)\|<\delta$, we have one of the following two cases:
1. $\|x(t)+y(t)\|<\epsilon,\ t\geqslant t_0+(n-1)[T_1(\epsilon)+T_2(\epsilon)]$ or
2. there exist $\xi_{5n-1}, \xi_{5n},\ \ t_0+(n-1)[T_1+T_2]<\xi_{5n-1}<\xi_{5n}<t_0+n[T_1+T_2]$, such that

$$V(\xi_{5n},x(\xi_{5n}),y(\xi_{5n}))\leqslant V(\xi_{5n-1},x(\xi_{5n-1}),y(\xi_{5n-1}))-c\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}.$$

If for any positive integer $n\geqslant v$ the second one holds, then by $\xi_{5(n-1)}<t_0+(n-1)[T_1+T_2]<\xi_{5n-1}$. Thus from ($H_5$) and (3.7) we obtain

$$\begin{aligned}
V(\xi_{5v},x(\xi_{5v}),y(\xi_{5v}))&\leqslant V(\xi_{5v-1},x(\xi_{5v-1}),y(\xi_{5v-1}))-c\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}\\
&\leqslant V(\xi_{5(v-1)},x(\xi_{5(v-1)}),y(\xi_{5(v-1)}))-c\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}\\
&\leqslant V(\xi_{5(v-1)-1},x(\xi_{5(v-1)-1}),y(\xi_{(5v-1)-1}))-c\left(\frac{1}{2}\delta_1\right)\frac{2\delta_1}{2L}\\
&\leqslant\cdots\\
&\leqslant V(\xi_4,x(\xi_4),y(\xi_4))-c\left(\frac{1}{2}\delta_1\right)\frac{(v-1)\delta_1}{2L}\\
&\leqslant b(M)-c\left(\frac{1}{2}\delta\right)\frac{v-1}{2L}\delta_1<0.
\end{aligned}\tag{3.12}$$

Then

$$\|x_0+y_0\|<\delta \text{ implies } \|x(t)+y(t)\|<\epsilon,\quad t\geqslant t_0+(n-1)[T_1(\epsilon)+T_2(\epsilon)].$$

This completes the proof of Theorem 4. □

References

[1] D.D. Bainov, P.S. Simeonov, Differentiability of solutions of systems with impulse effect with respect to initial data and parameter, Bull. Inst. Math. Acad. Sci. 15 (1987) 251–269.
[2] D.D. Bainov, P.S. Simeonov, Systems with impulse effect, in: Stability Theory and Applications, Ellis Horwood, Chichester, UK, 1989.

[3] G.K. Kulev, D.D. Bainov, On the asymptotic stability of systems with impulses by the direct method of Lyapunov, J. Math. Anal. Appl. 140 (1989) 324–340.
[4] V. Lakshmikantham, S. Leela, Cone-valued Lyapunov functions, Nonlinear Anal. Theory, Methods and Appl. 1 (3) (1977) 215–222.
[5] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[6] V. Lakshmikantham, S. Leela, Differential and integral inequalities, vol. I, New York, 1969.
[7] A.M. Somoilenko, N.A. Perslyuk, Differential Equations with Impulse Effect, Visca Skala Kiev, 1987.